An Overview on Behavioural Theory to Systems and Control

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Abstract

*This report deals with an overview of the behavioural theory based on the ' **Behaviors - an alternative approach to systems and control**' presented by Paula Rocha in the back to basics seminar in the University of Porto* [13]. *This report represents the topic/area of research, the main research questions, research challenges, main methodologies and the state of the art on the research on that topic presented in [13]. In this report, we extensively study the origin of behavioural theory and its scopes in the field of systems and control. We also investigate the recent works in this area.*

1. Introduction

In the recent years, behavioural theory plays a dominating role in the system and control which is now a very essential tool in the most of the engineering devices. The historical perspectives of control theory is more than 300 years older [6], the systematic progress and tremendous developments in this field occurred since fifty years ago which is now appeared as an independent field of research in system and control. The concept of optimal control came to light more than three century ago after the publication of Johann Bernouli's solution of the brachystochrone problem in 1697 [16]. Johann Bernouli had posed an open challenge to his contemporaries to solve the brachystochrone problem in 1696. Though the problem was solved by Newton and Bernouli independently, Bernouli was the first who articulated the principle of optimality [1]. Later on, various optimality principles were formulated by Pierre de Fermat (1601-1665) (in optics), Carl Friedrich Gauss (1777-1855), Jean d'Alembert (1717-1783), Pierre de Maupertuis (1698-1759), Euler, Lagrange and Hamilton, and Albert Einstein (1879-1955) (in mechanics). In 1957 Richard Bellman formulated the *dynamic programming* principle to the optimal control of discrete-time systems [2], and in 1958 Lev Pontryagin developed the *maximum*

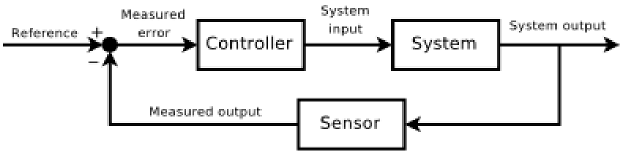

Fig. 1.1 The concept of the feedback loop to control the dynamic behavior of the system: this is negative feedback, because the sensed value is subtracted from the desired value to create the error signal which is amplified by the controller.

principle for solving nonlinear optimal control problems [10]. Both these optimality principles characterize the optimal control by means of a feedback law. The main idea of Bellman was to introduce the *value function* (the Bellman function) which satisfies the Hamilton-Jacobi equation. On the other hand, the Pontryagin maximum principle is based on maximization of the Hamiltonian associated to the system by means of the adjoint state equation.

In the 1960s, control was considered an electrical engineering subject, even though many applications of control involved mechanical machines or chemical processes. This could in part be explained by the fact that controllers were often implemented as electrical devices. But the mathematical methods used had a lot to do with it also as mentioned in [5].

The prevailing view of a dynamical system at that time in electrical engineering was input/output and frequency-domain based. Transfer functions were believed to be the way to characterize a system. Starting with Heaviside, symbolic calculus had been shown to be an effective tool for linear time-invariant dynamical systems. Under the influence of circuit theory, it had become evident that these methods allowed to analyze complex systems, by combining series, parallel, and feedback interconnections. The spirit of Heaviside's symbolic calculus was to be able to think of a differential operator or a delay as a formal indeterminate for which a differential operator or a delay can be substituted. Unfortunately, analysts had squeezed this marvelous idea in the mathematical rigor of Laplace transforms, using complex functions, with domains of convergence and other cumbersome but largely irrelevant mathematical traps.

Electrical engineers felt more comfortable with a view of a system as a frequency transformer than with any of the equivalent time-domain descriptions. The term filter, referring to the fact that a system passes some frequencies more easily that others, was synonymous for system. This view was completely prevailing in control, even more so than in the neighboring areas. In circuit theory there were, after all, many nonlinear devices. Frequency-domain analysis was not especially useful when thinking about Maxwell's equations, and information theory started from an altogether different set of principles. Computers were just around the corner, but they were viewed as calculating devices. The black-box approach was viewed as ideal for control. Transfer functions, applied almost uniquely to continuous- time single-input/single-output systems, was the mathematical language of control. A differential equation as $p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u$ with p and q real polynomials, was immediately transformed to a transfer function.

2. The Behavioural Approach

In the present century, the pioneer in the *behavioural systems theory* is J. C Willems. He has introduced the concept of *behavioural systems theory* in ([7], [8]).

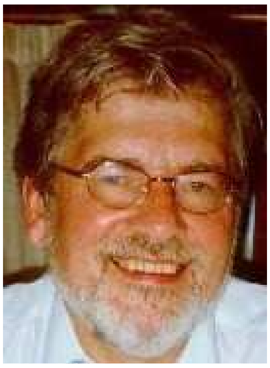

Fig. 2.1: J. C Willems: Pioneer archetype of the behavioural theory.

In [6], an extensive discussion on the modelling, analysis and control of linear time-invariant systems are presented. In this book, Willems showed that the two systems of differential equations are equivalent in the sense that they define the same behaviour. In [4], the behavioral approach is classified on the basis of the following premises.

► A mathematical model is a subset of a set of a priori possibilities. This subset is the behavior of the model. For a dynamical system, the behavior consists of the time trajectories that the model declares possible.

► The behavior is often given as a set of solutions of equations. Differential and difference equations are an effective, but highly nonunique, way of specifying the behaviour of a dynamical system.

► The behavior is the central concept in modeling. Equivalence of models, properties of models, model representations, and system identification must refer to the behavior.

► Both first principles models and models of interconnected systems usually contain latent variables in addition to the manifest variables that the model aims at. Elimination of latent variables compactifies the behavioral equations. For linear time-invariant differential systems, complete elimination of latent variables is possible.

► Physical systems are usually not endowed with a signal flow graph. Input/output models of physical systems are appropriate only in some special situations.

► Interconnected systems can be modelled using tearing, zooming, and linking. The interconnection architecture can be formalized as a graph with leaves. The nodes of the graph correspond to the subsystems, the edges correspond to the connected terminals, and the leaves correspond to terminals by which the interconnected system interacts with its environment. Interconnection of physical systems means variable sharing. Output-to-input assignment is often an unnecessary, inconvenient, and limiting way of viewing physical interconnections.

► System-theoretic concepts such as controllability and observability are simpler to define and more general in the behavioral setting than in the state-space setting. Controllability becomes a genuine property of a dynamical system rather than of just a state representation.

► Control means restricting the behavior of a plant by interconnection with a controller. Control by input selection, that is, open-loop control, and by feedback, that is, closed-loop control, are special cases.

► Linear time-invariant differential systems (including the special case of differential-algebraic systems) are in one-to-one correspondence with $R[\xi]$ sub-modules. This correspondence provides the ability to translate every property of a linear time-invariant differential behaviour into a property of the associated sub-module. Since these R[ξ] sub-modules are finitely generated, computer algebra-based algorithms can be used to analyze the system properties.

► For linear time-invariant differential systems, controllability is equivalent to the existence of an image representation, as well as to the case that the corresponding R[ξ] module is closed. Controllable linear time-invariant differential systems are in one-to-one correspondence with R(ξ)-subspaces.

► One-to-one correspondence of linear time-invariant systems with submodules, elimination of latent variables, and equivalence of controllability with the existence of an image representation are also valid for systems defined by constant-coefficient linear partial differential equations. The following figures gives a clear picture of interconnections.

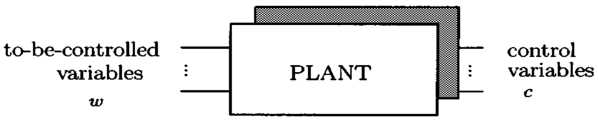

Fig. 2.2 Plant

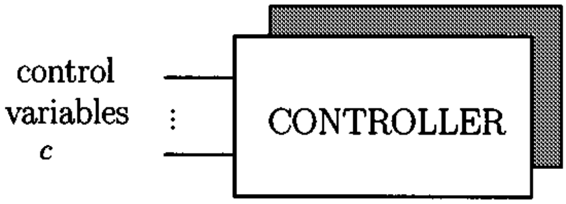

Fig. 2.3 Controller

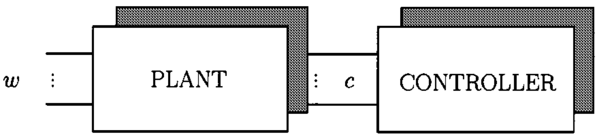

Fig. 2.4 The plant and controller after interconnection

3. Mathematical Models

Control theory is an interdisciplinary branch of engineering and mathematics, which deals with influencing the behavior of dynamical systems. The desired output of a system is called the *reference*. When one or more output variables of a system need to follow a certain reference over time, a controller manipulates the inputs to a system to obtain the desired effect on the output of the system. So, we will focus on dynamical systems because of their obvious importance in control problems with ***behavioral equations*** and ***latent variables*** as the important supporting characters. Mathematical models are expressed by ordinary or partial differential equations, they may involve the language of graphs or lattice diagrams, or require the notion of a transfer function or a formal language.

3.1 Universum and behaviour:

Assume that given a phenomenon which we want to model. To start with, we cast the situation in the language of mathematics by assuming that the ***phenomenon*** produces elements in a set which we will call the ***universum.*** Elements of $\mathbb{U}$ will be called the ***outcomes*** of the phenomenon. Now, a (deterministic) mathematical model for the phenomenon (viewed purely from the behavioural, the black box point of view) claims that certain outcomes are possible, while others are not. Hence a model recognizes a certain subset $\mathfrak{B}$ of $\mathbb{U}$. This subset will be called the ***behavior*** (of the model). Formally A ***mathematical model*** is a pair ($\mathbb{U}$, $\mathfrak{B}$) with $\mathbb{U}$ the ***universum its*** elements are called *outcomes* and $\mathfrak{B}$ the *behaviour.* A mathematical model can be illustrated by the following interesting example.

3.1.1 Example: During the ice age, shortly after Prometheus stole fire from the Gods, man realized that H_2O could appear, depending on the temperature, as water, steam, or ice. It took awhile longer before this situation was captured in a mathematical model. The generally accepted model takes the following form:

$\mathbb{U}$ = {ice, water, steam} × [-273, ∞)

and

$\mathfrak{B}$ = ({ice} × [-273,0]) U ({water} × [0,100]) U ({steam}× [100,∞)).

3.2 Behavioral Equations

Let $\mathbb{U}$ be a universum, $\mathbb{E}$ an abstract set, called the ***equating spaces,*** and let $f_1, f_2 : \mathbb{U} \to \mathbb{E}$. Then the mathematical model ($\mathbb{U}$, $\mathfrak{B}$) with $\mathfrak{B} = \{u \in \mathbb{U} : f_1(u) = f_2(u)\}$ is said to be described by **behavioural equation(s),** and will be denoted as ($\mathbb{U}$, $\mathbb{E}$, f_1, f_2) a behavioural equation representation. The best way of looking at the behavioral equations $f_1(u) = f_2(u)$ is as ***equilibrium conditions:*** the behaviour $\mathfrak{B}$ consists of those attributes for which two (sets of) quantities are in balance.

A few remarks are in order. First, in many applications models will be described by behavioural inequalities: simply take in the aforementioned definition Ito be an ordered space and consider the behavioural inequality $f_1(u) \le f_2(u)$. Many models in operations research (e.g., in linear programming) and in economics are of this nature. Second, every model can trivially be considered as coming from behavioural equations. Simply take $\mathbb{E}$ to be any set containing *0* and let *f* be any map from $\mathbb{U}$ to $\mathbb{E}$ such that $\mathfrak{B} = \{u \in \mathbb{U} : f(u) = 0\}$. Interesting representation questions occur when we want (f_1, f_2 *)* to have a concrete and appealing form or when we want the behavioural equations to be a set of equations, each allowing a natural and convincing interpretation, bringing into evidence such things as balance equations, conservation laws, simple constitutive laws, field equations, etc. Third, and most importantly, whereas equations uniquely specify the behaviour, the converse is obviously not true. Since we have a tendency to think of mathematical models in terms of equations, most models being presented in the form, it is important to emphasize their ancillary role: it is the behavior, the solution set of the behavioral equations, not the behavioral equations themselves, which is the essential result of a modeling procedure.

Many other properties of mathematical models can be nicely cast in the framework presented, for example, linearity, linearization, symmetry, variational principles, etc. In particular, a mathematical model ($\mathbb{U}$, $\mathfrak{B}$) is said to be ***linear*** if $\mathbb{U}$ is a vector space and $\mathfrak{B}$ is a linear subspace of $\mathbb{U}$.

Around 1960, the basic model for studying dynamics in control shifted from $p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u$ to $\dot{x} = f(x,u,t), \quad y = h(x,u,t)$. This was a major step forward. Multivariable systems could be covered without difficulty. Nonlinearities and time-variation could at least be put in evidence. Classical models from mechanics were a special case. With modest adaptations, finite state machines and automata were part of the same picture. So were, to some extent, systems described by PDE's. The input/state/output systems had much more modelling power and were far richer mathematically. By explicitly displaying its memory, the state, the model took into consideration initial conditions, something that transfer functions failed to do. The move to state space models constituted a true paradigm shift. The credit for this paradigm shift must go to scientists from the Soviet Union. Perhaps because physics, mechanics, and the calculus of variations were viewed as central, or perhaps because they were used to work with differential equations, but when Pontryagin cum suis started thinking about control, they chose $\dot{x} = f(x,u)$ as the model for articulating optimality.

In the late 1970s, Jan C. Willems [5] used a general 'set-theoretic' level, and end up with a detailed treatment of the highly structured linear time-invariant systems. He frowned on starting with the equations $\dot{x} = Ax + Bu,\ y = Cx + Du$ or even $\dot{x} = f(x,u),$ $y = h(x,u)$ with *(A, B,C,D)* such that $G(\xi) = D + C(I\xi - A)^{-1}B$. In this case there are actually many other ways of translating this module specification into dynamic equations. These were to be half-way points. Some reasoning should lead to the choice of u, y, and x.

3.3 Dynamical Systems

One of the fundamental tools of behavioural approach is dynamical systems as control theory deals with the behaviour of dynamical systems. Most dynamical systems are indeed described by behavioral equations. These are often differential or difference equations, sometimes integral equations. So, we will now apply the view of mathematical models in order to set up a language for dynamical systems. There have been many attempts to come up with a suitable axiomatic framework for the study of dynamical systems. In the study of dynamical systems, we are interested in situations where the events are maps from a set of time instances to a set of outcomes. The universum is then the collection of all maps from the set of independent variables to the set of dependent variables. In models of physical phenomena, it is customary to call the elements of the domain of a map independent variables and those of the codomain dependent variables. For dynamical systems, the independent variable is time, and the set of independent variables is therefore a subset of $\mathbb{R}$. In [4] the spatially distributed systems described by partial differential equations (PDEs), which involve multiple independent variables, reflecting, for example, time and space is discussed extensively. But now, we discuss only dynamical systems where $\mathbb{T}$ is a set of real numbers. The set of dependent variables $\mathbb{W}$ is the set in which the outcomes of the signals being modeled take on their values. We call $\mathbb{T}$ the *time axis* and $\mathbb{W}$ the *signal space*. Then a *dynamical system* is defined as a triple ($\mathbb{T}$, $\mathbb{W}$, $\mathfrak{B}$) in the following:

3.3.1 Definition: A dynamical system $\sum$ is given by a triple $\sum = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ with $\mathbb{T} \subseteq \mathbb{R}$ the *time axis,* $\mathbb{W}$ the *signal space,* and $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ the *behavior*. Thus, a dynamical system is defined by $\mathbb{T}$, the time instants of interest $\mathbb{W}$, the space in which the time signals which the system produces take on their values, and $\mathfrak{B}$, a family of $\mathbb{W}$-valued time trajectories. The sets $\mathbb{T}$ and $\mathbb{W}$ define the setting, the mathematization of the problem, while $\mathfrak{B}$ formalizes the laws which govern the system. According to the dynamical model $\sum$, time signals in $\mathfrak{B}$ can in principle occur, are compatible with the laws governing $\sum$, while those outside $\mathfrak{B}$ cannot occur, are prohibited.

In applications, elements of $\mathfrak{B}$ are required to be well-behaved maps from $\mathbb{T}$ to $\mathbb{W}$, at least measurable or locally integrable. In fact, when studying linear time-invariant differential system, we often assume for convenience of exposition that the elements of $\mathfrak{B}$ are infinitely differentiable. The behavior $\mathfrak{B}$ is the central object in this definition. The behavior formalizes which trajectories $w : \mathbb{T} \to \mathbb{W}$ are possible, according to the model. In the sequel, the terms "dynamical model," "dynamical system," and "behavior" are used as synonyms, since usually W and T follow from the context, leaving only $\mathfrak{B}$ as being specified by the model equations.

As an example, consider the motion of a planet around the sun. For this example, the time axis is $\mathbb{R}$ and the signal space is $\mathbb{R}^3$, since we are interested in describing the position trajectories that the planet can trace out. Before these motions were understood, every trajectory $w : \mathbb{R} \to \mathbb{R}^3$ could conceivably occur.

3.4 Linear time-invariant differential systems

We discuss the fundamentals of the theory of dynamical systems. We illustrate the use of the behavior to formulate system-theoretic concepts by means of two often used properties of dynamical systems, namely, linearity and time invariance.

3.4.1 Linearity: The dynamical system $\sum = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is *linear* if $\mathbb{W}$ is a vector space and $\mathfrak{B}$ a linear subspace of $\mathbb{W}^{\mathbb{T}}$, that is, if $w_1, w_2 \in \mathfrak{B}$ implies $\alpha w_1 + \beta w_2 \in \mathfrak{B}$ for all scalars α, β. Linearity means that superposition and scaling hold.

3.4.2 Time-invariant: The dynamical system $\sum = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is *time invariant* if $\mathbb{T}$ is closed under addition and $\sigma^t \mathfrak{B} \subseteq \mathfrak{B}$ for all $t \in \mathbb{T}$, where σ^t denotes the backward t-shift, defined by $(\sigma^t f)(t') := f(t' + t)$. Time invariance means that the shift of a legal trajectory is again legal.

3.4.3 Differential: *differential*, meaning consists of the solutions of a system of differential equations. For some model classes, for example, for linear time invariant differential systems, there always exists a componentwise input/output partition of the system variables. Consider the differential equation,

$$R_0 w + R_1 \frac{d}{dt} w + R_2 \frac{d^2}{dt^2} w + \ldots\ldots\ldots + R_n \frac{d^n}{dt^n} w = 0,$$

with $R_0, R_1, R_2, \ldots\ldots, R_n \in \mathbb{R}^{g \times w}$.

Combined with the polynomial matrix, $R(\xi) = R_0 + R_1(\xi) + \ldots\ldots\ldots + R_n(\xi^n)$, where the real matrices $R_0, R_1, \ldots, R_n$ are the parameters of the model, and the differential equation specifies which time trajectories $w : \mathbb{R} \to \mathbb{R}^{w}$ belong to the behavior. This is in the mercifully short notation can be written as,

$$R\left(\frac{d}{dt}\right) w = 0.$$

3.5 Controllability and Observability

Another essential tools in behavioural model are controllability and observability. Controllability and observability are dual aspects of the same problem. The introduction of controllability is one of the milestones in the history of control. This notion played a seminal role in the early development of the state-space theory of dynamical systems. Since then, controllability features as a regularizing assumption in essentially every major theoretical development in the field. By now, controllability is one of the first things taught in introductory control courses. A state-space system is *controllable* if, for any two states, there is an input that drives the system from the first state to the second.

A good overview of the early history of controllability is given in the lecture notes [15] by Rudy Kalman. After discussing the state-space notions of controllability, observability, and minimal state representation, it is stated that

´Only much later did it become clear, however, that a dynamical system is always controllable if it is derived from an external description’ [15, p. 136].

The traditional definition of controllability emerged at the Modern Theory of Control and refers to the ca - pational system, controlled, be able to transfer state between any 2 points. In other words, a system is controllable if, for any two states, ex - istira a control system that transfers the first es - wards the second in finite time. The controllability, defined in this classic format, presents some drawback. From the outset, the controllability assumes that the system is arranged in the form input / state, which requires the selection of the early states of the model. Additionally, the situation of uncontrollability may be clearer if this factor is due to the mistaken choice states or if the variables manipulated by the controller no influence on the total system. For this reason, the pro - ownership of controllability refers only to repre - tion of states of the model and may not be generalizable for the overall system [4].

A system is said to be *observable* if, for any possible sequence of state and control vectors, the current state can be determined in finite time using only the outputs (this definition is slanted towards the state space representation). Less formally, this means that from the system's outputs it is possible to determine the behaviour of the entire system. If a system is not observable, this means the current values of some of its states cannot be determined through output sensors: this implies that their value is unknown to the controller and, consequently, that it will be unable to fulfil the control specifications referred to these outputs.

For time-invariant linear systems in the state space representation, a convenient test to check if a system is observable exists. Consider a SISO system with n states, if the rank of the following *observability matrix*

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is equal to n, then the system is observable. The rationale for this test is that if n rows are linearly independent, then each of the n states is viewable through linear combinations of the output variables $y(k)$.

4. Recent Developments

In the last few decades tremendous achievements have been occurred in the behavioural models in systems and control theory. In addition to Jan C. Willems, several authors have made a significant contribution in the aforesaid field. Belur and Trentelman [3] studied the control by interconnection in a behavioral framework. In particular, for linear differential systems with two types of variables, to-be-controlled variables and control variables, they established the necessary and sufficient conditions for regular implementability of a given sub-behavior of the manifest plant behavior. They formulated the pole placement problem and the stabilization problem as problems of finding suitable, regularly implementable sub-behaviors. These formulations were completely representation-free. Using the characterization of regular implementability, they obtained necessary and sufficient conditions for pole placement and stabilization. Again, these conditions were expressed in terms of properties of the plant behavior itself, and not as properties of a particular representation of it. As an illustration, They studied the case that the plant is given in an input–state–output representation. They proved that the controlled behaviors obtained in the pole placement problem and the stabilization problem can, in fact, be implemented by means of (singular) feedback. In fact, if for the plant to be controlled an actuator–sensor structure is specified in advance, then a feedback controller can be found that respects this actuator–sensor structure. Finally, we have established the connection between freedom of disturbances in the controlled system, and regularity of interconnections. In a minicourse on behavioral systems theory [11], Rapisarda and Willems (2006) presented an effective algebraic representation of bilinear and quadratic functional of the system variables and their derivatives. Working with functionals at most natural level, two-variables polynomial representation; operations/properties in time domain; algebraic operations; differentiation, integration, positivity; Lyapunov theory, dissipativity, model reduction by balancing. Rapisarda in [12] has developed the concepts of bilinear- and quadratic differential forms used in modeling the control problems which are functional of the system variables and their derivatives arising in optimal control and in the theory of Lagrangian or Hamiltonian mechanics as well as in Lyapunov stability theory and also showed a couple of applications to system- and control theory problems such as the modelling of linear Hamiltonian systems, and an equipartition of energy principle. In 2007. Kojima *et. al.* defined [9] the canonical representative for the equivalence class consisting of all polynomial- and quadratic differential operators that take the same values on a given subspace of C. One of the significant contributors in the area of behavioural modeling in systems and control is P. Rocha. Several achievements have already by P. Rocha specially in the Quaternionic behaviours, Periodic behaviours and *2D* behaviours [13].

The Control of multidimensional *nD* behaviors are also proposed in [14]. She investigated the three types of stabilizability defined for *nD* systems within the behavioural framework, namely trajectory stabilizability, set-controllability to a stable behaviour and stabilizability by interconnection and proved that stabilizability by interconnection is the strongest than the others. Researches are going on for the further achievements in the aforesaid fields of systems and control.

4. Concluding Remarks

The field of systems and control has come a long way in the last 50 years. The mathematical methods used have expanded enormously. The techniques that have been developed for trajectory transfer, stabilization, disturbance attenuation, observers, adaptation, and robustness are deep and relevant. The modeling ideas ranging from state space systems to model reduction and uncertainty modeling are rich and versatile. The paradigm of open systems, combined with interconnection, make it into an area that fits modern technological developments well, even though systems and control has benefited less from the explosion of numerically driven and microprocessor or internet based applications than some neighboring areas, as signal processing, communication, and optimization.

In this paper, we have outlined the basic concepts underlying the dynamical systems framework which have been developed over the last few years. The fundamental ideas center around the notions of the ***behavior*** of a dynamical system, of ***behavioral equations,*** typically, difference or differential equations specifying the behavior (in a many-to-one way), and finally, ***latent variables*** which will almost always be present in models obtained from first principles. ***State variables*** may be viewed as an especially important class of latent variables. The framework presented provides an appealing and natural setting in which many classical concepts and problems attain new meaning and content: linearity, time invariance, controllability, observability, etc. It also leads to a multitude of representation questions, in particular input-output, state, and inputstate/output representation problems. Underlying these representation questions there are invariably parametrization problems. For the class of systems studied here these are typically polynomial matrix or matrix parametrizations.

We have extensively investigated the origin of behavioral approaches in the systems and control applied in many engineering devices. We have also studied the recent developments in behavioral theory and also the framework presented here is especially well suited for treating questions of modeling, for example system identification and model approximation problems. We hope this study in the behavioral theory will open a new horizon the near future.

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